FORCE ACTING ON A SPHERE IN AN INHOMOGENEOUS FLOW OF AN IDEAL INCOMPRESSIBLE FLUID

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UDC 532.529.6

We consider the motion of a sphere of variable radius in an inhomogeneous (with gradient velocity) and unsteady potential flow of an ideal incompressible fluid. Assuming that the sphere radius is small in comparison with the distance to the flow boundary, we calculate the force of the effect of the flow on the sphere.

Certain cases of motion of a sphere in an arbitrary potential flow were investigated by Zhukovskii [1]. For a stationary elliptical cylinder the problem was solved by Gurevich [2]; the force acting on a moving circular cylinder of variable radius was calculated by Yakimov [3]. A similar problem was considered in [4] for a sphere, but an incorrect expression was obtained for the force.

The derivation is based, just as in [4], on the direct integration of the pressure forces over the sphere, but instead of a model problem on the flow potential about the sphere in a second-order multipole field we consider the problem of the velocity potential of an arbitrary flow, and we estimate the error in the equations obtained for the forces.

1. Velocity Potential

The velocity potential at the point $y_i = q_i$ (the y_i are the Cartesian coordinates, and the q_i are the coordinates of the center of the sphere) in the absence of the sphere is represented by the Taylor series

$$\Phi_0 = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \Phi_0}{\partial y_i \partial y_j \ldots \partial y_k}(\mathbf{q})\, x_i x_j \ldots x_k \quad (x_i = y_i - q_i) \tag{1.1}$$

Here and below, we sum over the repeated indices i, j, ...,k, each of which assumes the values 1, 2, and 3, the number of the indices i, j, ... k in (1.1) equaling n. Each term of the series n = 1, 2, ... in (1.1) is a harmonic function.

If a sphere of radius R is located in the flow, then for the velocity potential the following condition is satisfied:

$$\frac{\partial \Phi}{\partial r} = \frac{q_i^{\cdot} x_i}{R} + R^{\cdot} \quad \text{for} \quad r = R \quad (r^2 = x_i x_i) \tag{1.2}$$

If we neglect the effect of the far boundaries, then we must require that the perturbation of potential $\Phi - \Phi_0$ have singularities only within the sphere and decrease out to infinity. The unique harmonic function that satisfies these conditions and condition (1.2) will be

$$\Phi = -R^2 R^{\cdot}/r - q_i^{\cdot} x_i R^3/2r^3 + \sum_{n=0}^{\infty} \frac{1}{n!}\left(1 + \frac{n}{n+1}\frac{R^{2n+1}}{r^{2n+1}}\right) \frac{\partial^n \Phi_0}{\partial y_i \partial y_j \ldots \partial y_k}(\mathbf{q})\, x_i x_j \ldots x_k \tag{1.3}$$

This equation holds if the distance r_0 to the flow boundaries considerably exceeds the sphere radius. Otherwise we must take into account an additional potential perturbation owing to the promixity of the sphere to the flow boundaries. Since the velocity perturbation at the boundaries caused by a distant body is of order R^3/r_0^3 for R = const and R^2/r_0^2 for $R^{\cdot} \neq 0$, the perturbation of potential (1.3) for $R \to 0$ will be of order no

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 182-184, July-August, 1973. Original article submitted November 24, 1972.

less than R^3 or R^2, respectively. We can conclude that only the first several terms of the series (1.3) need be taken into account.

2. Force of Effect of Flow on Sphere

Based on the potential (1.3), applying the Cauchy–Lagrange integral, we can obtain values for the pressure on the sphere S, and calculate the force of the effect **F.**

The Cauchy–Lagrange integral in a system moving with velocity $q_i^{\cdot}$ relative to this system, in which the motion of the fluid is described by the potential Φ, has the form

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - q_i^{\cdot}\frac{\partial\Phi}{\partial x_i} + \frac{p}{\rho} - U = f(t)$$

Here U is the potential of the external mass forces g_i.

The force of the effect of the flow on the sphere equals

$$F_m = \rho\int_S\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - q_i^{\cdot}\frac{\partial\Phi}{\partial x_i} - U\right)\frac{x_m}{R}\,dS \qquad (m = 1, 2, 3) \tag{2.1}$$

On the sphere, the terms of the series (1.3) are orthogonal to x_m for $n > 1$. Therefore

$$\int_S \frac{\partial\Phi}{\partial t}\frac{x_m}{R}\,dS = \frac{4\pi R^3}{3}\frac{dv_m}{dt} + \frac{2\pi}{3}\frac{d}{dt}R^3(v_m - q_m^{\cdot}) \qquad (v_m = v_m(q, t) = \partial\Phi_0/\partial y_m) \tag{2.2}$$

Here v_m is the flow velocity in the absence of the sphere. To calculate the contribution of the remaining terms to Eq. (2.1) we must have in the integrand the components of the flow velocity on the sphere. From (1.3) for $r = R$

$$\begin{aligned}\frac{\partial\Phi}{\partial x_l} &= \frac{R^{\cdot}}{R}x_l - \frac{q_l^{\cdot}}{2} + \frac{3}{2}q_j^{\cdot}\frac{x_jx_l}{R^2}\\ &+ \sum_{n=1}^{\infty}\frac{n(2n+1)}{(n+1)!}P_{il}\frac{\partial^n\Phi_0}{\partial y_i\partial y_j\ldots\partial y_k}(\mathbf{q})\,x_j\ldots x_k\\ P_{il} &= \delta_{il} - x_ix_l/R^2\end{aligned} \tag{2.3}$$

On the surface of the sphere

$$P_{il}P_{ls} = P_{is}$$

Taking account of this equation and Eq. (2.3) we obtain

$$\begin{aligned}{}^1\!/_2(\nabla\Phi)^2 - \dot{\mathbf{q}}\nabla\Phi &= {}^1\!/_2(R^{\cdot 2} - \mathbf{q}^{\cdot 2} + P_{is}w_iw_s)\\ w_i &= \frac{3}{2}(v_i - q_i^{\cdot}) + \sum_{n=2}^{\infty}\frac{n(2n+1)}{(n+1)!}\frac{\partial^n\Phi_0}{\partial y_i\partial y_j\ldots\partial y_k}(\mathbf{q})\,x_j\ldots x_k\end{aligned} \tag{2.4}$$

A contribution to the integral (2.1) is made only by polynomials with an odd sum of powers x_1, x_2, and x_3, appearing in the term $P_{is}w_iw_s$, which figures in (2.4). Dropping terms of order R^5 and above, we can write

$$\int_S\left[\frac{1}{2}(\nabla\Phi)^2 - q^{\cdot}\nabla\Phi\right]\frac{x_m}{R}\,dS = \frac{5}{2}(q_l^{\cdot} - v_l)\frac{\partial v_k}{\partial y_j}\int_S P_{lk}\frac{x_jx_m}{R}\,dS \tag{2.5}$$

We then must take into account that

$$\int_S x_kx_lx_jx_m\,dS = (\delta_{kj}\delta_{lm} + \delta_{kl}\delta_{jm} + \delta_{km}\delta_{lj})\frac{4\pi}{15}R^4$$

Hence also from the determination of P_{lk} in (2.3) we obtain

$$\int_S P_{lk} \frac{x_j x_m}{R} dS = (\delta_{kj}\delta_{lm} + \delta_{lj}\delta_{km} - 4\delta_{jm}\delta_{kl}) \frac{4\pi}{15} R^3$$

Substitution of the last equation into (2.5) gives

$$\int_S \left[\frac{1}{2}(\nabla\Phi)^2 - \mathbf{q}^{\cdot}\nabla\Phi\right] \frac{x_m}{R} dS = -2\pi R^3 (q_k^{\cdot} - v_k) \frac{\partial v_m}{\partial y_k} \tag{2.6}$$

From Eqs. (2.1),(2.2), and (2.6) we can obtain for the force of the effect of the flow on the body

$$\frac{F_m}{2\pi\rho R^3} = \frac{\partial v_m}{\partial t} + v_k \frac{\partial v_m}{\partial y_k} - \frac{1}{3} q_m^{\cdot\cdot} + \frac{(R^3)^{\cdot}}{3R^3} (v_m - \dot{q}_m) - \frac{2}{3} g_m \tag{2.7}$$

Here v_m is the flow velocity in the absence of the sphere; the values of v_m and grad v_m, and also the mass force g_m, are calculated at the point q_i. As follows from the derivation, the error of Eq. (2.7) is of order R^5. For $R^{\cdot} \neq 0$, owing to the effect of the flow boundaries, the error can be $\sim R^{\cdot 2} R^4$.

According to Eq. (2.7) the force of the effect of the flow on a sphere of constant volume is determined by the acceleration of a fluid particle in the absence of the sphere, and by the acceleration of the sphere, and it depends neither on the velocity of the sphere in the flow nor on the velocity of the fluid.

Equation (2.7) agrees with Zhukovskii's equation [1]

$$\mathbf{F} = {}^3/_2 \rho V \mathbf{a}$$

obtained for a stationary sphere in a flow with potential equal to the product of a function of the coordinates and a function of the time (**a** is the absolute acceleration of the fluid at the point coinciding with the center of the sphere, and V is the volume of the sphere).

For a gas bubble in the absence of viscosity we can set **F** = 0, since the mass of the gas is negligibly small. Then (2.7) gives that the acceleration of the gas bubble is equal to three times the acceleration of the fluid. This fact was known earlier for a homogeneous flow (without a velocity gradient) [5].

The author thanks G. Yu. Stepanov for helpful remarks.

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